Lecture 33

Last Time

<u>Def</u>: A non-zero vec $\vec{v} \in \mathbb{R}^n$ is an eigenvector for $A \in M_{n \times n}$ with <u>eigenvalue</u> $\lambda \in \mathbb{R}$

if $A\vec{v} = \lambda \vec{v}$

- A is <u>diagonalizable</u> i.f.f. $\exists$ basis for $\mathbb{R}^n$ consisting e-vecs $\vec{v}_1, \dots, \vec{v}_n$ for A.

On Exam: No defs on eigenvectors/eigenvalues

↳ e-vecs/e-vals prolly only computational.

- $\lambda \in \mathbb{R}$ is an e-val of A i.f.f. $\det(A - \lambda I) = 0$
- then $\vec{v} \in \mathbb{R}^n$ is an e-vec with e-val λ i.f.f. $\vec{v} \in \ker(A - \lambda I)$.

$\boxed{\text{Ex}}$

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

Find evals/e-vecs of A

$$\det(A-\lambda I) = 0 \quad = \begin{bmatrix} 3-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & -1-\lambda \end{bmatrix} \rightarrow (3-\lambda)(2-\lambda)(-1-\lambda) = 0$$

$\lambda = 3, 2, -1$

<u>Moral</u>: If A is upper triangular, the e-vals of A are diagonal entries of A.

e-vals	e-vecs
3	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
2	$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$
-1	$\begin{bmatrix} 1 \\ -4 \\ 12 \end{bmatrix}$

$\lambda = 3$

$$[A-3I \mid 0] \rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \vec{v} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$\lambda = 2$

$$[A-2I \mid 0] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_2 is free so

$$\vec{V} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$\lambda = -1$

$$[A-(-1)I \mid 0] \rightarrow \left[\begin{array}{ccc|c} 4 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 12 & 0 & -1 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_3 is free

so $x_3 \begin{bmatrix} 1/12 \\ -1/3 \\ 1 \end{bmatrix}$

you can scale by 12

to get $\begin{bmatrix} 1 \\ -4 \\ 12 \end{bmatrix}$

$A = S \Lambda S^{-1}$ $\quad S = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -4 \\ 0 & 0 & 12 \end{bmatrix}$ $\leftarrow$ each column is an e-vec

$\Lambda = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ $\leftarrow$ notice each entry is the e-val corresponding to the col in S.

Proposition: A matrix $A \in M_{n \times n}$ is diagonalizable if it has eigenvectors $\vec{v}_1, \dots, \vec{v}_n$ with distinct e-vals $\lambda_1, \dots, \lambda_n \in \mathbb{R}$

Follows from

Lem: If $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$ are e-vecs for A with distinct e-vals then $\vec{v}_1, \dots, \vec{v}_k$ are LI.

Pf of Lem: Suppose not, I.e. $\exists$ non-trivial relation

$$\vec{0} = c_1 \vec{v}_1 + \dots + c_k \vec{v}_k$$

$$\rightarrow \quad \begin{array}{r} \vec{0} = A\vec{0} = c_1 \lambda_1 \vec{v}_1 + \dots + c_k \lambda_k \vec{v}_k \\ - \quad \vec{0} = \lambda_k \vec{0} = c_1 \lambda_k \vec{v}_1 + \dots + c_k \lambda_k \vec{v}_k \\ \hline \vec{0} = c_1(\lambda_1 - \lambda_k)\vec{v}_1 + \dots + c_{k-1}(\lambda_{k-1} - \lambda_k)\vec{v}_{k-1} \end{array}$$

$\rightarrow$ non-trivial relation among $\vec{v}_1, \dots, \vec{v}_k$

$\hookrightarrow$ because $v_k \neq \vec{0}$ and $\lambda_i \neq \lambda_k$ when $i \neq k$.

$\rightarrow \dots \rightarrow \vec{0} = \tilde{c}_1 \vec{v}_1$

but this is impossible because $\tilde{c}_1 \neq 0$ and $\vec{v}_1 \neq 0$

$\therefore$ $\vec{v}_1, \dots, \vec{v}_k$ are independent $\blacksquare$

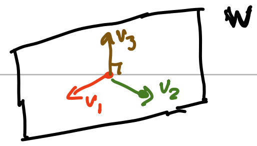
$\boxed{\text{Ex}}$ (not covered by Prop) : $A = [\text{proj}_W]$ where $W \subset \mathbb{R}^3$ is a plane through $\vec{0}$. Then

A is similar to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \Lambda$

$Av_1 = v_1$

$Av_2 = v_2$

$Av_3 = 0$



- λ = e-val of $A \in M_{n \times n} \longleftrightarrow \underbrace{\det(A - \lambda I)}_{P(\lambda)} = 0$

"characteristic polynomial of A"

- Obs : $P(\lambda)$ = poly of degree n

<u>Def</u> : Let $\lambda \in \mathbb{R}$ be an eigenvalue of $A \in M_{n \times n}$

- The <u>algebraic multiplicity</u> of λ is its multiplicity as a root of

$$\det(A - \lambda I) = (\lambda - \lambda_1)^k \cdot \text{stuff}$$

- The <u>geometric multiplicity</u> of λ_1 is $\dim \ker(A - \lambda_1 I)$